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THE PHASE-AMPLITUDE APPROACH TO NONLOCAL SCATTERING

by



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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FALL, 1970

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and
recommend to the Faculty of Graduate Studies for
acceptance, a thesis entitled THE PHASE-AMPLITUDE
APPROACH TO NONLOCAL SCATTERING, submitted by Walter
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ments for the degree of Master of Science.

ABSTRACT

The phase-amplitude approach is considered as a means of calculating the regular and irregular wave functions for nonlocal scattering. The equivalent local wave function and potential may then be expressed exactly in terms of these. A separable kernel is used as an example. A class of nonlocal kernels that can be obtained from the Green's functions for second-order differential equations is considered. The momentum dependent potential is derived and is expanded in terms of the index of nonlocality to obtain the effective mass approximation. A specific example, the nonlocal square well, is solved analytically and its equivalent local wave function and potential are examined for the s-state.

ACKNOWLEDGMENTS

I wish to sincerely thank Dr. M. Razavy, my supervisor, for suggesting this problem and for his guidance in the major part of this work. It was unfortunate that Dr. Razavy was unable to be present for the completion of this work, consequently I am indebted to Dr. Y. Takahashi for his time and help in preparing the final draft of this thesis.

Further, I would like to express my appreciation for the financial assistance of the National Research Council via a Postgraduate Scholarship.

Finally, I am grateful to Mrs. Mary Yiu for typing this thesis.

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CHAPTER 1

NONLOCAL INTERACTIONS

I. Introduction

Historically, the use of nonlocal interactions began with the advent of extensive research into the properties of nuclear matter. Basically, the interaction between complex nuclei is a many-body problem and complex potentials can be calculated, at least in principle, by summing over the two-body nucleon-nucleon forces. This fundamental approach has been developed by Brueckner⁽¹⁾ and others⁽²⁾, and has gone a long way in justifying the success of the phenomenological optical model potential that is used in the description of nucleon-nucleus interactions. The theory showed that additional features should be incorporated into the phenomenological model. It turns out that the nucleon-nucleus interaction operator $\langle \underline{r} | v | \underline{r}' \rangle$, for finite nuclei⁽¹⁾ is nonlocal in coordinate space. This means that the potential acting on a particle centered at a point \underline{r} depends not only on \underline{r} , but also on the value of the wave function throughout all the region of nonlocality and so takes account of the finite size of the incident particle and the dispersive nature of the nuclear matter. With the interaction now

described by a matrix $\langle \underline{r} | V | \underline{r}' \rangle \equiv V(\underline{r}, \underline{r}')$, the term $V(\underline{r}) \psi(\underline{r})$ in the one particle Schrödinger equation that normally describes local interactions now becomes

$$\int V(\underline{r}, \underline{r}') \psi(\underline{r}') d^3 \underline{r}'$$

where $V(\underline{r}, \underline{r}')$ is the nonlocal potential.

Very little is known about the form of this potential. Obviously it complicates matters because now the wave function satisfies an integro-differential equation.

Brown and De Dominicis⁽³⁾ have shown that $V(\underline{r}, \underline{r}')$ must be symmetric in \underline{r} and \underline{r}' if the nucleon eigen-functions are to be orthogonal to each other within the nucleus. Furthermore, the short range of the nuclear forces and the calculations of Brueckner and his coworkers⁽¹⁾ suggest that the range of nonlocality is small. If this is the case, then by introducing a range of nonlocality $\frac{1}{\beta}$ as a parameter, one may be able to describe the effects of the nonlocality in a phenomenological treatment without knowing the detailed structure of the interaction matrix. Of course one must require that the potential becomes local, $V(\underline{r}, \underline{r}') = V(\underline{r}) \delta(\underline{r} - \underline{r}')$, as the nonlocal range goes to zero ($\beta \rightarrow \infty$).

Nonlocal potentials are not confined only to the many-body problem of nucleon-nucleus scattering.

Giltinan and Thaler⁽⁴⁾ have demonstrated that an examination of the energy dependence of the 1S_0 and 1D_2 phase shifts for proton-proton scattering below 340 Mev., requires that the singlet even parity nucleon-nucleon interaction be nonlocal. This is not surprising, for one might expect that the exchange of many particles and resonances between two nucleons will produce non-local forces.

Any nonlocal potential that is used to describe a scattering problem will have its energy dependent local counterpart. Indeed, there is no reason to suppose that the nonlocal potential itself does not change with energy. However until such a time as field theorists or S-matricists can derive a nonlocal potential from more basic principles, a reasonable approach would be to keep the nonlocal potential energy independent and vary it in an appropriate way with the least number of parameters so that the equivalent local potential changes with energy in a natural way consistent with scattering data.

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In Section II we discuss the momentum equivalence of the nonlocal potential and obtain the nonlocal radial Schrödinger wave equation.

The phase-amplitude method is considered in Chapter 2 as a means for obtaining the nonlocal wave function.

We extend this method in Chapter 3 to also calculate the "irregular" nonlocal wave function. The equivalent local wave function and potential may then be expressed exactly in terms of these. A separable kernel is used as an example.

A class of nonlocal kernels that can be obtained from the Green's functions for second-order differential equations is considered in Chapter 4. The momentum dependent potential is derived and is expanded in terms of the nonlocality parameter to obtain the effective mass approximation. A specific example, the nonlocal square well, is solved analytically and its equivalent local wave function and potential are examined for the s-state.

II. Nonlocal Schrödinger Wave Equation

The nonlocal Schrödinger equation for scattering of two particles of equal mass M by a nonlocal potential $v(\underline{r}, \underline{r}')$ is

$$(\nabla^2 + k^2) \psi(\underline{r}) = \frac{M}{\hbar^2} \int v(\underline{r}, \underline{r}') \psi(\underline{r}') d^3 \underline{r}' . \quad (1-1)$$

For simplicity we assume the particles are spinless and uncharged. Furthermore if we restrict ourselves to elastic scattering, the nonlocal potential is real. The wave function and nonlocal potential may be expanded

in terms of spherical harmonics

$$\psi(\underline{r}) = \sum_{\ell, m} \frac{u_{\ell}(\underline{r})}{\underline{r}} Y_{\ell m}(\theta, \phi) \quad (1-2)$$

$$\begin{aligned} \frac{M}{\hbar^2} V(\underline{r}, \underline{r}') &= \sum_{\ell} \frac{2\ell+1}{4\pi} \frac{K_{\ell}(\underline{r}, \underline{r}')}{\underline{r}\underline{r}'} P_{\ell} \left(\frac{\underline{r} \cdot \underline{r}'}{\underline{r}\underline{r}'} \right) \\ &= \sum_{\ell, m} \frac{K_{\ell}(\underline{r}, \underline{r}')}{\underline{r}\underline{r}'} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') . \end{aligned} \quad (1-3)$$

The spherical harmonics satisfy the differential equation

$$\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\} Y_{\ell m}(\theta, \phi) = -\ell(\ell+1) Y_{\ell m}(\theta, \phi)$$

and have the orthonormality conditions

$$\int_{\Omega} Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) d\Omega = \delta_{\ell\ell'} \delta_{mm'} .$$

Using these we can substitute eq. (1-2) and eq. (1-3) into eq. (1-1) to obtain

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} u_{\ell}(r) = \int_0^{\infty} K_{\ell}(r, r') u_{\ell}(r') dr' . \quad (1-4)$$

The kernel $K_{\ell}(r, r')$ is calculated from $V(\underline{r}, \underline{r}')$ by inverting eq. (1-3)

$$K_\ell(\underline{r}, \underline{r}') = 2\pi r r' \left(\frac{M}{\hbar^2}\right)^{+1} \int_{-1}^1 V(\underline{r}, \underline{r}') P_\ell(\cos\theta) d(\cos\theta). \quad (1-5)$$

Mathematically, a nonlocal potential is equivalent to a momentum dependent potential. This can best be illustrated by expanding $\psi(\underline{r}')$ about $\underline{r}' = \underline{r}$ in a Taylor's series.

$$\begin{aligned} \psi(\underline{r}') &= \psi(\underline{r} + \underline{r}' - \underline{r}) \\ &= \sum_n \frac{1}{n!} [(\underline{r}' - \underline{r}) \cdot \nabla]^n \psi(\underline{r}) \\ &= \exp [(\underline{r}' - \underline{r}) \cdot \nabla] \psi(\underline{r}) \quad . \end{aligned} \quad (1-6)$$

If we now replace ∇ by the momentum operator $\frac{i}{\hbar} \underline{p}$ and substitute eq. (1-6) into eq. (1-1), we have

$$(\nabla^2 + k^2) \psi(\underline{r}) = U(\underline{r}, \underline{p}) \psi(\underline{r}) \quad (1-7)$$

where $U(\underline{r}, \underline{p})$ is the momentum dependent potential given by

$$U(\underline{r}, \underline{p}) = \int V(\underline{r}, \underline{r}') e^{\frac{i}{\hbar} (\underline{r}' - \underline{r}) \cdot \underline{p}} d^3 \underline{r}' \quad (1-8)$$

CHAPTER 2

THE NONLOCAL PHASE AND AMPLITUDE FUNCTIONS

I. Phase function

In Chapter 1, we obtained the radial wave equation for nonlocal scattering

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} u_\ell(r) = \int_0^\infty K_\ell(r,s) u_\ell(s) ds. \quad (2-1)$$

We assume the usual conditions on the nonlocal kernel asymptotically and at the origin. These imply the existence of a regular solution $u_\ell(r)$ characterized by the boundary conditions

$$u_\ell(0) = 0 \quad (2-2)$$

$$u_\ell(r) \xrightarrow{r \rightarrow \infty} \text{const.} \sin \left[kr + \delta_\ell - \frac{\ell\pi}{2} \right]. \quad (2-3)$$

The constants δ_ℓ are the scattering phase shifts and are defined only mod (2π) .

Following Calogero⁽⁵⁾, we outline a formalism whereby the phase shifts are calculated as the limits of phase functions for large r . For local potentials, this method transforms the second order differential wave equation with two-point boundary conditions into a set of

two coupled first order nonlinear differential equations with initial value conditions. This procedure greatly simplifies the study of the analytical properties and numerical calculation of the phase shifts. For nonlocal potentials the generalization involves coupled integral equations.

For convenience we define

$$F_\ell(r) = \int_0^\infty K_\ell(r, s) u_\ell(s) ds . \quad (2-4)$$

With the aid of the boundary condition at the origin given by eq. (2-2), the nonlocal wave equation may be transformed into the integral equation

$$u_\ell(r) = \hat{j}_\ell(kr) - \frac{1}{k} \int_0^r [\hat{j}_\ell(kr) \hat{\eta}_\ell(ks) - \hat{\eta}_\ell(kr) \hat{j}_\ell(ks)] F_\ell(s) ds . \quad (2-5)$$

The Riccati-Bessel functions $\hat{j}_\ell(z)$ and $\hat{\eta}_\ell(z)$ are defined by

$$\hat{j}_\ell(z) = z j_\ell(z) \quad (2-6)$$

$$\hat{\eta}_\ell(z) = z \eta_\ell(z) ,$$

where $j_\ell(z)$ and $\eta_\ell(z)$ are the spherical Bessel functions of the first and second kind respectively⁽⁶⁾. We introduce the auxiliary functions $c_\ell(r)$ and $s_\ell(r)$ by

$$u_\ell(r) = c_\ell(r) \hat{j}_\ell(kr) - s_\ell(r) \hat{n}_\ell(kr) . \quad (2-7)$$

Comparison with the integral equation for $u_\ell(r)$ implies

$$s_\ell(r) = - \frac{1}{k} \int_0^r \hat{j}_\ell(ks) F_\ell(s) ds \quad (2-8)$$

$$c_\ell(r) = 1 - \frac{1}{k} \int_0^r \hat{n}_\ell(ks) F_\ell(s) ds .$$

Thus, differentiating with respect to r , we have

$$s'_\ell(r) = - \frac{1}{k} \hat{j}_\ell(kr) F_\ell(r) \quad (2-9)$$

$$c'_\ell(r) = - \frac{1}{k} \hat{n}_\ell(kr) F_\ell(r) ,$$

with the initial value conditions

$$s_\ell(0) = 0 \quad (2-10)$$

$$c_\ell(0) = 1 .$$

From eq. (2-9) we have the identity

$$c'_\ell(r) \hat{j}_\ell(kr) - s'_\ell(r) \hat{n}_\ell(kr) = 0 . \quad (2-11)$$

Using this and the property that the Riccati-Bessel functions satisfy

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} \begin{bmatrix} \hat{j}_\ell(kr) \\ \hat{n}_\ell(kr) \end{bmatrix} = 0 , \quad (2-12)$$

we differentiate eq. (2-7) twice to find

$$u_{\ell}''(r) + (k^2 - \frac{\ell(\ell+1)}{r^2}) u_{\ell}(r) = c_{\ell}'(r) \hat{j}_{\ell}'(kr) - s_{\ell}'(r) \hat{n}_{\ell}'(kr). \quad (2-13)$$

However, the right hand side of this expression is just equal to $F_{\ell}(r)$ by comparison with eq. (2-1). If we now multiply eq. (2-13) by $u_{\ell}(r)$, we obtain

$$-k[s_{\ell}'(r)c_{\ell}(r) - s_{\ell}(r)c_{\ell}'(r)] = F_{\ell}(r)u_{\ell}(r), \quad (2-14)$$

where we have made use of eq. (2-7) and the Wronskian identity

$$\hat{j}_{\ell}(kr) \hat{n}_{\ell}'(kr) - \hat{n}_{\ell}(kr) \hat{j}_{\ell}'(kr) = k. \quad (2-15)$$

Dividing eq. (2-14) by $u_{\ell}^2(r)$ yields

$$\left[\frac{s_{\ell}(r)}{c_{\ell}(r)} \right]' = -\frac{1}{k} \left[\hat{j}_{\ell}(kr) - \frac{s_{\ell}(r)}{c_{\ell}(r)} \hat{n}_{\ell}(kr) \right]^2 \frac{F_{\ell}(r)}{u_{\ell}(r)}. \quad (2-16)$$

We define the phase function $\delta_{\ell}(r)$ by

$$\tan \delta_{\ell}(r) = \frac{s_{\ell}(r)}{c_{\ell}(r)}, \quad (2-17)$$

to obtain

$$\delta_{\ell}'(r) = -\frac{1}{k} \left[\hat{j}_{\ell}(kr) \cos \delta_{\ell}(r) - \hat{n}_{\ell}(kr) \sin \delta_{\ell}(r) \right]^2 \frac{F_{\ell}(r)}{u_{\ell}(r)}, \quad (2-18)$$

subject to the initial value condition,

$$\delta_\ell'(0) = 0 .$$

The significance of the phase function is illustrated by expressing the wave function in terms of it, that is;

$$u_\ell(r) = \frac{c_\ell(r)}{\cos\delta_\ell(r)} [\cos\delta_\ell(r) \hat{j}_\ell(kr) - \sin\delta_\ell(r) \hat{n}_\ell(kr)] . \quad (2-19)$$

Since the Riccati-Bessel functions have the asymptotic behavior

$$\hat{j}_\ell(kr) \xrightarrow[r \rightarrow \infty]{} \sin [kr - \frac{\ell\pi}{2}] \quad (2-20)$$

$$\hat{n}_\ell(kr) \xrightarrow[r \rightarrow \infty]{} -\cos [kr - \frac{\ell\pi}{2}] ,$$

then we have

$$u_\ell(r) \xrightarrow[r \rightarrow \infty]{} \frac{c_\ell(r)}{\cos\delta_\ell(r)} \sin [kr - \frac{\ell\pi}{2} + \delta_\ell(r)] . \quad (2-21)$$

If we compare this with the condition expressed by eq. (2-3), we see that the scattering phase shifts are the asymptotic limits of the phase function.

$$\delta_\ell = \lim_{r \rightarrow \infty} \delta_\ell(r) . \quad (2-22)$$

II. Amplitude function

The amplitude function $\alpha_\ell(r)$ is defined by

$$\alpha_\ell(r) = c_\ell(r)/\cos\delta_\ell(r) , \quad (2-23)$$

and is more conveniently expressed in the form

$$\alpha_\ell^2(r) = s_\ell^2(r) + c_\ell^2(r) . \quad (2-24)$$

Differentiating this equation and using eq. (2-9) we find

$$\alpha_\ell'(r) = - \frac{F_\ell(r)}{k} [\hat{j}_\ell(kr)\sin\delta_\ell(r) + \hat{n}_\ell(kr)\cos\delta_\ell(r)] . \quad (2-25)$$

If we use eq. (2-18), this can be rearranged in terms of the phase function

$$\alpha_\ell'(r) = \alpha_\ell(r) \delta_\ell'(r) \left\{ \frac{\hat{j}_\ell(kr)\sin\delta_\ell(r) + \hat{n}_\ell(kr)\cos\delta_\ell(r)}{\hat{j}_\ell(kr)\cos\delta_\ell(r) - \hat{n}_\ell(kr)\sin\delta_\ell(r)} \right\} \quad (2-26)$$

with the initial value condition

$$\alpha_\ell(0) = 1 . \quad (2-27)$$

The differential equations for the phase and amplitude functions may be cast into a more convenient form if we make use of the Riccati-Bessel phase and amplitude functions $\hat{D}_\ell(z)$ and $\hat{\delta}_\ell(z)$ defined by

$$\hat{j}_\ell(z) = \hat{D}_\ell(z) \sin \hat{\delta}_\ell(z) \quad (2-28)$$

$$\hat{n}_\ell(z) = - \hat{D}_\ell(z) \cos \hat{\delta}_\ell(z) .$$

Hence eqs. (2-19), (2-18), (2-26) become

$$u_\ell(r) = \alpha_\ell(r) \hat{D}_\ell(kr) \sin [\hat{\delta}_\ell(kr) + \delta_\ell(r)] \quad (2-29)$$

$$\delta_\ell'(r) = - \frac{1}{k} \hat{D}_\ell^2(kr) \sin^2 [\hat{\delta}_\ell(kr) + \delta_\ell(r)] \cdot \frac{F_\ell(r)}{u_\ell(r)} \quad (2-30)$$

$$\alpha_\ell'(r) = -\alpha_\ell(r) \delta_\ell'(r) \cotg [\hat{\delta}_\ell(kr) + \delta_\ell(r)] . \quad (2-31)$$

For the s-state ($\ell=0$), the above equations simplify to

$$\begin{aligned} u(r) &= \alpha(r) \sin [kr + \delta(r)] \\ \delta'(r) &= - \frac{1}{k} \sin^2 [kr + \delta(r)] \frac{F(r)}{u(r)} \\ \alpha'(r) &= - \alpha(r) \cotg [kr + \delta(r)] . \end{aligned} \quad (2-32)$$

Furthermore, if the potential is local

$$\text{i.e. } K(r, s) = V(r) \delta(r-s) ,$$

then the s-state phase function satisfies the first-order nonlinear differential equation

$$\delta'(r) = - \frac{V(r)}{k} \sin^2 [kr + \delta(r)] \quad (2-33)$$

with $\delta(0) = 0$.

Analytical solutions to the above equation (and in general for any ℓ wave) are of course confined to potentials for which the Schrödinger equation is a priori solvable. However, numerical solutions may now be obtained in a direct and rapid manner for any physical potential.

III. Integro-differential equation for $\delta_\ell(r)$

If the nonlocal phase function is known, the amplitude function is determined from eq. (2-31) by integration. Using the condition $\alpha_\ell(0)=1$, we have

$$\alpha_\ell(r) = \exp \left\{ - \int_0^r \delta'_\ell(s) \cotg[\hat{\delta}_\ell(ks) + \delta_\ell(s)] ds \right\} . \quad (2-34)$$

Using the definition of $F_\ell(r)$ in eq. (2-4) we now have

$$\frac{F_\ell(r)}{\alpha_\ell(r)} = \int_0^\infty ds K_\ell(r, s) \hat{D}_\ell(ks) \sin[\hat{\delta}_\ell(ks) + \delta_\ell(s)] .$$

$$\cdot \exp \left\{ \int_s^r \delta'_\ell(t) \cotg[\hat{\delta}_\ell(kt) + \delta_\ell(t)] dt \right\} . \quad (2-35)$$

The integrand of the exponential term is not singular, because the poles of $\cotg[\hat{\delta}_\ell(kt) + \delta_\ell(t)]$ are cancelled by zeros of $\delta'_\ell(t)$. The phase function may now be expressed as the solution of the integro-differential equation

$$\begin{aligned}
 \delta'_\ell(r) &= -\frac{1}{k} \hat{D}_\ell(kr) \sin[\hat{\delta}_\ell(kr) + \delta_\ell(r)] \\
 &\cdot \int_0^\infty K_\ell(r, s) \hat{D}_\ell(ks) \sin[\hat{\delta}_\ell(ks) + \delta_\ell(s)] ds \\
 &\cdot \exp\left\{\int_s^r \delta'_\ell(t) \cotg[\hat{\delta}_\ell(kt) + \delta_\ell(t)] dt\right\}. \quad (2-36)
 \end{aligned}$$

IV. Born and improved Born Approximation

Except for special forms of the nonlocal kernel, the integro-differential equation is probably impossible to solve. One must resort to approximation methods to evaluate the phase function for arbitrary potential. The most straightforward of these is the method of successive approximations. That is; the successively better approximations are given by $\delta_\ell^{(n+1)}(r)$ where

$$\begin{aligned}
 \delta_\ell^{(n+1)}(r) &= -\frac{1}{k} \hat{D}_\ell(kr) \sin[\hat{\delta}_\ell(kr) + \delta_\ell^{(n)}(r)] \\
 &\cdot \int_0^\infty K_\ell(r, s) \hat{D}_\ell(ks) \sin[\hat{\delta}_\ell(ks) + \delta_\ell^{(n)}(s)] ds \\
 &\cdot \exp\left\{\int_s^r \delta_\ell^{(n)}(t) \cotg[\hat{\delta}_\ell(kt) + \delta_\ell^{(n)}(t)] dt\right\}. \quad (2-37)
 \end{aligned}$$

To start the iteration we use $\delta^{(0)}(r) = 0$. To first order we have

$$\delta'_\ell(r) \approx -\frac{1}{k} \hat{j}_\ell(kr) \int_0^\infty K_\ell(r, s) \hat{j}_\ell(ks) ds \quad (2-38)$$

and the phase shifts are therefore

$$\delta_\ell \approx -\frac{1}{k} \int_0^\infty \int_0^\infty K_\ell(r, s) \hat{j}_\ell(kr) \hat{j}_\ell(ks) dr ds. \quad (2-39)$$

However, these are just the phase shifts in the Born approximation, for if we consider the eq. (2-16) for the tangent of the phase function

$$[\tan \delta_\ell(r)]' = -\frac{1}{k} [\hat{j}_\ell(kr) - \tan \delta_\ell(r) \hat{n}_\ell(kr)]^2 \frac{F_\ell(r)}{u_\ell(r)} \quad (2-40)$$

and take as zero-order approximation $\delta_\ell(r) = 0$, then to first order

$$\tan \delta_\ell \equiv -\frac{1}{k} \int_0^\infty \int_0^\infty K_\ell(r, s) \hat{j}_\ell(kr) \hat{j}_\ell(ks) dr ds. \quad (2-41)$$

This reassures us that the validity of the Born approximation requires that the phase shifts be small, so that the tangent does not differ appreciably from its argument.

In any case, one should be able to show that the iteration scheme outlined for the phase function converges to the correct solution for any physical kernel. However, because of the strong nonlinearity of the phase equation, the resulting expressions already become unmanageably complicated by the second cycle. One must ultimately

resort to numerical calculation to use the successive approximations scheme.

The Born approximation can be improved upon by making the approximations

$$[\hat{j}_\ell(kr) - \tan\delta_\ell(r) \hat{n}_\ell(kr)]^2 \approx \hat{j}_\ell^2(kr) - 2 \tan\delta_\ell(r) \hat{n}_\ell(kr) \hat{j}_\ell(kr) \quad (2-42)$$

$$u_\ell(r) \approx \hat{j}_\ell(kr) ,$$

thereby effectively linearizing the equation for $\tan\delta_\ell(r)$

$$[\tan\delta_\ell(r)]' \approx -\frac{1}{k} [\hat{j}_\ell(kr) - 2 \tan\delta_\ell(r) \hat{n}_\ell(kr)] \cdot \int_0^\infty K_\ell(r, s) \hat{j}_\ell(ks) ds . \quad (2-43)$$

The solution to this equation is given by

$$\begin{aligned} \tan\delta_\ell(r) = & -\frac{1}{k} \int_0^r \hat{j}_\ell(ks) ds \int_0^\infty K_\ell(s, t) \hat{j}_\ell(kt) dt \\ & \cdot e^{\frac{2}{k} \int_s^r \hat{n}_\ell(kx) dx} \int_0^\infty K_\ell(x, y) \hat{j}_\ell(ky) dy \end{aligned} \quad (2-44)$$

and the resulting phase shifts are determined by taking the limit.

V. Separable kernel

One example of a nonlocal potential where the phase and amplitude functions can be determined analytically is that of a separable kernel,

$$K_\ell(r, s) = g_\ell(r) g_\ell(s) . \quad (2-45)$$

From eq. (2-4) we have

$$F_\ell(r) = \frac{k}{A_\ell} g_\ell(r) \quad (2-46)$$

where the constants A_ℓ are defined by,

$$A_\ell = k \int_0^\infty g_\ell(s) u_\ell(s) ds . \quad (2-47)$$

Eqs. (2-8), (2-17) and (2-24) then yield

$$\tan \delta_\ell(r) = \frac{- \int_0^r \hat{j}_\ell(ks) g_\ell(s) ds}{A_\ell - \int_0^r \hat{n}_\ell(ks) g_\ell(s) ds} \quad (2-48)$$

and

$$\begin{aligned} \alpha_\ell^2(r) &= 1 - \frac{2}{A_\ell} \int_0^r \hat{n}_\ell(ks) g_\ell(s) ds + \\ &+ \frac{1}{A_\ell^2} \left[\left(\int_0^r \hat{n}_\ell(ks) g_\ell(s) ds \right)^2 + \left(\int_0^r \hat{j}_\ell(ks) g_\ell(s) ds \right)^2 \right] \end{aligned} \quad (2-49)$$

The constants A_ℓ are obtained by making use of the integral equation for the wave function (eq. (2-5)) in eq. (2-41)

$$A_\ell = \frac{k + \int_0^\infty g_\ell(s) ds \int_0^s \{ \hat{j}_\ell(ks) \hat{\eta}_\ell(kt) - \hat{\eta}_\ell(ks) \hat{j}_\ell(kt) \} g_\ell(t) dt}{\int_0^\infty g_\ell(s) \hat{j}_\ell(ks) ds}.$$

(2-50)

CHAPTER 3

EQUIVALENT LOCAL POTENTIALS

The use of nonlocal potentials in nuclear physics makes a detailed study of their connection with local potentials worthwhile. We define an equivalent local potential (ELP) by the requirement that it reproduces the same asymptotic wave function as the nonlocal potential. Mathematically we require real functions $x_\ell(r)$ and $V_\ell(r)$ that satisfy the equations

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} x_\ell(r) = V_\ell(r) x_\ell(r) \quad (3-1)$$

$$x_\ell(r) \xrightarrow[r \rightarrow \infty]{} \sin \left[kr - \frac{\ell\pi}{2} + \delta_\ell \right] . \quad (3-2)$$

Here $V_\ell(r)$ is the ELP to be determined, $x_\ell(r)$ is the regular equivalent local wave function and δ_ℓ are the phase shifts determined from the asymptotic behavior of the nonlocal wave function $u_\ell(r)$.

It is immediately obvious from eq. (2-1) that trivial solutions are

$$x_\ell(r) = \text{const.} u_\ell(r)$$

$$V_\ell(r) = \frac{\int_0^\infty K_\ell(r, s) u_\ell(s) ds}{u_\ell(r)} . \quad (3-3)$$

However, in this case, $V_\ell(r)$ has a pole whenever $u_\ell(r)$ has a zero and is therefore far from satisfactory.

Ideally we require an ELP that has no poles (unless perhaps at the origin) and falls away rapidly for a short range kernel. We now outline an analysis due to Fiedeldey⁽⁷⁾ for obtaining ELP's that satisfy these requirements, although they will be implicit functions of the scattering energy and angular momentum.

I. Equivalent Local Potentials expressed in terms of nonlocal wave functions

We assume that there exist two linearly independent solutions of the equivalent local wave equation (3-1), a regular solution $x_\ell(r)$ satisfying eq. (3-2), and an irregular solution $y_\ell(r)$, that has the asymptotic behavior

$$y_\ell(r) \xrightarrow[r \rightarrow \infty]{} \cos \left[kr - \frac{\ell\pi}{2} + \delta_\ell \right] \quad (3-4)$$

and whose "local Wronskian" with $x_\ell(r)$ is

$$x'_\ell(r) y_\ell(r) - x_\ell(r) y'_\ell(r) = k \quad . \quad (3-5)$$

We shall relate the nonlocal wave function $u_\ell(r)$ to the equivalent local one by an interpolating function $f_\ell(r)$,

$$u_\ell(r) = f_\ell(r) x_\ell(r) \quad . \quad (3-6)$$

Substituting eq. (3-6) into the nonlocal wave equation (2-1), we find

$$\begin{aligned} \left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} x_\ell(r) = \\ \frac{1}{f_\ell(r)} \left\{ \int_0^\infty K_\ell(r, s) f_\ell(s) x_\ell(s) ds - f_\ell''(r) x_\ell(r) - 2f_\ell'(r) x_\ell'(r) \right\}, \end{aligned} \quad (3-7)$$

and from eq. (3-1), the ELP must be

$$v_\ell(r) = \frac{\int_0^\infty K_\ell(r, s) f_\ell(s) x_\ell(s) ds - f_\ell''(r) x_\ell(r) - 2f_\ell'(r) x_\ell'(r)}{f_\ell(r) x_\ell(r)}. \quad (3-8)$$

With the aid of the local Wronskian (eq. (3-5)), we introduce the identity

$$\begin{aligned} kx_\ell(s) &= [x_\ell'(r) y_\ell(s) - x_\ell(s) y_\ell'(r)] x_\ell(r) \\ &+ [y_\ell(r) x_\ell(s) - x_\ell(r) y_\ell(s)] x_\ell'(r). \end{aligned} \quad (3-9)$$

The ELP then becomes

$$\begin{aligned} v_\ell(r) &= \frac{1}{f_\ell(r)} \left\{ \frac{1}{k} \int_0^\infty K_\ell(r, s) f_\ell(s) [x_\ell'(r) y_\ell(s) - x_\ell(s) y_\ell'(r)] ds - f_\ell''(r) \right. \\ &+ \left. \frac{1}{f_\ell(r)} \left(\frac{x_\ell'(r)}{x_\ell(r)} \right) \left\{ \frac{1}{k} \int_0^\infty K_\ell(r, s) f_\ell(s) [y_\ell(r) x_\ell(s) - x_\ell(r) y_\ell(s)] ds - 2f_\ell'(r) \right\} \right\} \end{aligned} \quad (3-10)$$

We now try to choose $f_\ell(r)$ in such a way that there are no singularities in the ELP. If we assume that $f_\ell(r) > 0$, then the absence of poles is ensured by requiring that

$$f_\ell'(r) = \frac{1}{2k} \int_0^\infty K_\ell(r, s) f_\ell(s) [y_\ell(r) x_\ell(s) - x_\ell(r) y_\ell(s)] ds , \quad (3-11)$$

and it follows that

$$v_\ell(r) = - \frac{f_\ell''(r)}{f_\ell(r)} + \frac{1}{k f_\ell(r)}$$

$$\int_0^\infty K_\ell(r, s) f_\ell(s) [x_\ell'(r) y_\ell(s) - x_\ell(s) y_\ell'(r)] ds . \quad (3-12)$$

In the limiting case of a local kernel, $K_\ell(r, s) = V(r) \delta(r-s)$, eqs. (3-11) and (3-12) must reduce to

$$f_\ell(r) = \text{constant} \quad (3-13)$$

$$v_\ell(r) = V(r) .$$

With the aid of eq.(3-5) it is easily seen that these conditions are fulfilled. Equation (3-11) is particularly useful in establishing that the function $v_\ell(r)$ defined by

$$v_\ell(r) = f_\ell(r) y_\ell(r) \quad (3-14)$$

also satisfies the nonlocal wave equation. This may be easily proven by considering the "nonlocal Wronskian"

$$u_\ell'(r) v_\ell(r) - u_\ell(r) v_\ell'(r) = k f_\ell^2(r) \quad (3-15)$$

which is derived from the definitions supplied by eqs. (3-6) and (3-14) and with the aid of eq. (3-5). If we differentiate eq. (3-15) and use eq. (3-11) to replace $f_\ell'(r)$ it follows that

$$u_\ell''(r) v_\ell(r) - u_\ell(r) v_\ell''(r) = \int_0^\infty K_\ell(r,s) u_\ell(s) ds - u_\ell(r) \int_0^\infty K_\ell(r,s) v_\ell(s) ds. \quad (3-16)$$

If we now use the nonlocal wave equation (eq. (2-1)) to replace $u_\ell''(r)$, it follows that $v_\ell(r)$ satisfies the equation

$$v_\ell''(r) + \{k^2 - \frac{\ell(\ell+1)}{r^2}\} v_\ell(r) = \int_0^\infty K_\ell(r,s) v_\ell(s) ds. \quad (3-17)$$

However, because of the normalization imposed on $y_\ell(r)$ by eq. (3-5), the "irregular" solution to the nonlocal wave equation, $v_\ell(r)$, will have the asymptotic behavior

$$v_\ell(r) \xrightarrow{r \rightarrow \infty} f_\ell(\infty) \cos \left[kr - \frac{\ell\pi}{2} + \delta_\ell \right], \quad (3-18)$$

where $f_\ell(\infty)$ and δ_ℓ are now determined from the asymptotic behavior of the regular nonlocal wave function $u_\ell(r)$, i.e.

$$u_\ell(r) \xrightarrow{r \rightarrow \infty} f_\ell(\infty) \sin \left[kr - \frac{\ell\pi}{2} + \delta_\ell \right] . \quad (3-19)$$

Assuming that we have a priori determined $u_\ell(r)$ and hence $v_\ell(r)$ according to the prescription above, the interpolating function, $f_\ell(r)$ and the ELP, $v_\ell(r)$, can now be obtained exactly in terms of them. $f_\ell(r)$ is of course determined from the nonlocal Wronskian (eq. (3-15)). The expression for the ELP may be simplified using the equation

$$x_\ell'(r) = \frac{u_\ell'(r)}{f_\ell(r)} - \frac{u_\ell(r) f_\ell'(r)}{f_\ell^2(r)}$$

and a similar equation for $y_\ell'(r)$. We have then

$$\begin{aligned} v_\ell(r) = & - \frac{f_\ell''(r)}{f_\ell(r)} + \frac{1}{kf_\ell^2(r)} \int_0^\infty K_\ell(r,s) [u_\ell'(r)v_\ell(s) - v_\ell'(r)u_\ell(s)] ds \\ & + \frac{1}{kf_\ell^2(r)} \left[\frac{f_\ell'(r)}{f_\ell(r)} \right] \int_0^\infty K_\ell(r,s) [v_\ell(r)u_\ell(s) - u_\ell(r)v_\ell(s)] ds . \end{aligned} \quad (3-20)$$

However, from eq. (3-16), we note that the integral in the above equation is just the differential of the nonlocal Wronskian (eq. (3-15)). Thus we have

$$\begin{aligned}
 v_\ell(r) = & -\frac{f_\ell''(r)}{f_\ell(r)} + 2 \left(\frac{f_\ell'(r)}{f_\ell(r)} \right)^2 + \\
 & + \frac{1}{k f_\ell^2(r)} \int_0^\infty K_\ell(r, s) [u_\ell'(r)v_\ell(s) - v_\ell'(r)u_\ell(s)] ds . \tag{3-21}
 \end{aligned}$$

A simpler expression for the ELP can be obtained by considering, as solutions for the equivalent local wave equation, the following,

$$x_\ell(r) = A_\ell(r) \sin B_\ell(r) \tag{3-22}$$

$$y_\ell(r) = A_\ell(r) \cos B_\ell(r)$$

where

$$A_\ell(r) \xrightarrow{r \rightarrow \infty} 1 \tag{3-23}$$

$$B_\ell(r) \xrightarrow{r \rightarrow \infty} kr - \frac{\ell\pi}{2} + \delta_\ell .$$

If we now substitute for $x_\ell(r)$ in eq. (3-1) we obtain for the ELP,

$$\begin{aligned}
 v_\ell(r) = & k^2 - \frac{\ell(\ell+1)}{r^2} + \frac{A_\ell''(r)}{A_\ell(r)} - [B_\ell'(r)]^2 + \\
 & + B_\ell'(r) \left(\frac{2A_\ell'(r)}{A_\ell(r)} + \frac{B_\ell''(r)}{B_\ell'(r)} \right) \cotg B_\ell(r) . \tag{3-24}
 \end{aligned}$$

A similar ELP is obtained for $y_\ell(r)$ except that $\cotg B_\ell(r)$ is replaced by $-\tan B_\ell(r)$. It is obvious that to have

both $x_\ell(r)$ and $y_\ell(r)$ satisfying the same equation, and to avoid untold numbers of singularities due to the tangent and cotangent functions, we must have

$$\frac{2 A_\ell'(r)}{A_\ell(r)} + \frac{B_\ell''(r)}{B_\ell'(r)} = 0 \quad . \quad (3-25)$$

However, this condition is already implied by the Wronskian prescription in eq. (3-5). That is, we have

$$A_\ell^2(r) B_\ell'(r) = k \quad (3-26)$$

and eq. (3-25) follows by differentiation. The ELP becomes

$$v_\ell(r) = k^2 - \frac{\ell(\ell+1)}{r^2} + \frac{A_\ell''(r)}{A_\ell(r)} - \frac{k^2}{A_\ell^4(r)} \quad (3-27)$$

where, from eq. (3-22)

$$A_\ell^2(r) = x_\ell^2(r) + y_\ell^2(r) \quad .$$

In terms of the nonlocal wave functions we have

$$A_\ell^2(r) = \frac{u_\ell^2(r) + v_\ell^2(r)}{f_\ell^2(r)} \quad . \quad (3-28)$$

It is easily verified that with $A_\ell(r)$ as defined above, the ELP given by eq. (3-21) reduces to eq. (3-27) as it must. Equation (3-27) for the ELP is not very useful physically since it seems to indicate that the potential

depends explicitly on the scattering energy and angular momentum. This is not the case, for eqs. (3-15) and (3-21) for $f_\ell(r)$ and $v_\ell(r)$, show that the range for which they differ from a constant and zero respectively, coincides with that of the nonlocal potential.

II. The irregular nonlocal wave function

Since we require the irregular nonlocal wave function to calculate the ELP, we shall now show how Calogero's phase-amplitude method can be extended to calculate $v_\ell(r)$.

The integral equation for $v_\ell(r)$ can be written as

$$v_\ell(r) = a_\ell \hat{j}_\ell(kr) - b_\ell \hat{\eta}_\ell(kr) - \frac{1}{k} \int_0^r [\hat{j}_\ell(kr) \hat{\eta}_\ell(ks) - \hat{j}_\ell(ks) \hat{\eta}_\ell(kr)] \cdot G_\ell(s) ds \quad (3-29)$$

where a_ℓ and b_ℓ are constants to be determined and $G_\ell(r)$ is defined by

$$G_\ell(r) = \int_0^\infty K_\ell(r, s) v_\ell(s) ds . \quad (3-30)$$

As in Chapter 2, we define the auxiliary functions

$$s_\ell(r) = -a_\ell + \frac{1}{k} \int_0^r \hat{\eta}_\ell(ks) G_\ell(s) ds \quad (3-31)$$

$$c_\ell(r) = b_\ell - \frac{1}{k} \int_0^r \hat{j}_\ell(ks) G_\ell(s) ds$$

so that eq. (3-29) may now be written as

$$v_\ell(r) = -c_\ell(r)\hat{\eta}_\ell(kr) - s_\ell(r)\hat{j}_\ell(kr) . \quad (3-32)$$

The functions $s_\ell(r)$ and $c_\ell(r)$ here defined have no connection with the ones used in Chapter 2. Since their use is only auxiliary, we shall have no conflict of definitions.

From eqs. (3-31) we have

$$s'_\ell(r) = \frac{1}{k} \hat{\eta}_\ell(kr) G_\ell(r) \quad (3-33)$$

$$c'_\ell(r) = -\frac{1}{k} \hat{j}_\ell(kr) G_\ell(r) ,$$

and hence the identity

$$s'_\ell(r) \hat{j}_\ell(kr) + c'_\ell(r) \hat{\eta}_\ell(kr) = 0 . \quad (3-34)$$

Using this identity and eq. (2-12), we differentiate eq. (3-32) twice to find

$$v''_\ell(r) + \left\{ k^2 - \frac{\ell(\ell+1)}{r^2} \right\} u_\ell(r) = -c'_\ell(r)\hat{\eta}'_\ell(kr) - s'_\ell(r)\hat{j}'_\ell(kr) , \quad (3-35)$$

and therefore by comparison with eq. (3-17)

$$G_\ell(r) = -c'_\ell(r) \hat{\eta}'_\ell(kr) - s'_\ell(r) \hat{j}'_\ell(kr) . \quad (3-36)$$

With the aid of eqs. (2-15), (3-32), and (3-34), the above equation may be written as

$$-k \left(\frac{s_\ell(r)}{c_\ell(r)} \right)' = \left[-\hat{n}_\ell(kr) - \left(\frac{s_\ell(r)}{c_\ell(r)} \right) \hat{j}_\ell(kr) \right]^2 \frac{G_\ell(r)}{v_\ell(r)} . \quad (3-37)$$

Defining the "irregular phase function" $\gamma_\ell(r)$ by

$$\tan \gamma_\ell(r) = \frac{s_\ell(r)}{c_\ell(r)} , \quad (3-38)$$

eq. (3-37) becomes

$$\gamma_\ell'(r) = -\frac{1}{k} \left[\hat{n}_\ell(kr) \cos \gamma_\ell(r) + \hat{j}_\ell(kr) \sin \gamma_\ell(r) \right]^2 \frac{G_\ell(r)}{v_\ell(r)} . \quad (3-39)$$

The irregular wave function is now written as

$$v_\ell(r) = \frac{c_\ell(r)}{\cos \gamma_\ell(r)} \left[-\hat{n}_\ell(kr) \cos \gamma_\ell(r) - \hat{j}_\ell(kr) \sin \gamma_\ell(r) \right] , \quad (3-40)$$

and hence we define the "irregular amplitude function"

$\beta_\ell(r)$ by

$$\beta_\ell(r) = c_\ell(r) / \cos \gamma_\ell(r) ,$$

which is more conveniently written as

$$\beta_\ell^2(r) = c_\ell^2(r) + s_\ell^2(r) . \quad (3-41)$$

We differentiate this expression and use eq. (3-33) to obtain

$$\beta_\ell'(r) = \frac{1}{k} \left[\hat{n}_\ell(kr) \sin \delta_\ell(r) - \hat{j}_\ell(kr) \cos \delta_\ell(r) \right] G_\ell(r) . \quad (3-42)$$

Eq. (3-39) is now used to replace $G_\ell(r)$ in eq. (3-42)

$$\beta'_\ell(r) = \beta_\ell(r) \gamma'_\ell(r) \begin{Bmatrix} \hat{j}_\ell(kr) \cos \gamma_\ell(r) - \hat{\eta}_\ell(kr) \sin \gamma_\ell(r) \\ -\hat{\eta}_\ell(kr) \cos \gamma_\ell(r) - \hat{j}_\ell(kr) \sin \gamma_\ell(r) \end{Bmatrix}. \quad (3-43)$$

We shall now make use of the Riccati-Bessel phase and amplitude functions defined by eq. (2-28) to rewrite eqs. (3-39), (3-40), and (3-43).

$$v_\ell(r) = \beta_\ell(r) \hat{D}_\ell(kr) \cos [\hat{\delta}_\ell(kr) + \gamma_\ell(r)] \quad (3-44)$$

$$\gamma'_\ell(r) = -\frac{1}{k} \hat{D}_\ell^2(kr) \cos^2 [\hat{\delta}_\ell(kr) + \gamma_\ell(r)] \left(\frac{G_\ell(r)}{v_\ell(r)} \right) \quad (3-45)$$

$$\beta'_\ell(r) = \beta_\ell(r) \gamma'_\ell(r) \tan [\hat{\delta}_\ell(kr) + \gamma_\ell(r)]. \quad (3-46)$$

These equations are very similar in form to their counterparts (eqs. (2-29) to (2-31)) obtained for the regular nonlocal wave function. There is the important difference that instead of initial value conditions as for $\alpha_\ell(r)$ and $\delta_\ell(r)$ we now have end value conditions on $\beta_\ell(r)$ and $\gamma_\ell(r)$. To see this more clearly we make use of the asymptotic expansions of $\hat{\delta}_\ell(kr)$ and $\hat{D}_\ell(kr)$ which may be determined from eqs. (2-20) and (2-28).

$$\hat{D}_\ell(kr) \xrightarrow[r \rightarrow \infty]{} 1 \quad (3-47)$$

$$\hat{\delta}_\ell(kr) \xrightarrow[r \rightarrow \infty]{} kr - \frac{\ell\pi}{2}$$

Therefore the irregular wave function has the asymptotic behavior

$$v_\ell(r) \xrightarrow[r \rightarrow \infty]{} \beta_\ell(\infty) \cos\left[kr - \frac{\ell\pi}{2} + \gamma_\ell(\infty)\right] \quad (3-48)$$

and according to eqs. (3-18) and (3-19) we must therefore have the end value conditions

$$\beta_\ell(r) \xrightarrow[r \rightarrow \infty]{} \alpha_\ell(\infty) \quad (3-49)$$

$$\gamma_\ell(r) \xrightarrow[r \rightarrow \infty]{} \delta_\ell$$

where

$$\alpha_\ell(\infty) = \lim_{r \rightarrow \infty} \alpha_\ell(r) \quad (3-50)$$

$$\delta_\ell = \lim_{r \rightarrow \infty} \delta_\ell(r) .$$

If one attempts a numerical calculation, the sequence of steps is quite clear. One integrates the equations ((2.29)-(2.31)) to some point, \bar{r} say, where the kernel is effectively zero and obtains $\alpha_\ell(\bar{r})$ and $\delta_\ell(\bar{r})$. Then using the end value conditions $\beta_\ell(\bar{r}) = \alpha_\ell(\bar{r})$, and $\gamma_\ell(\bar{r}) = \delta_\ell(\bar{r})$, one integrates the equations ((3-44)-(3.46)) backwards to the origin to evaluate $v_\ell(r)$.

Because of the structural similarity of the phase-amplitude equations for the regular and irregular non-local wave functions one can obtain a similar form of integro-differential equation for $\gamma_\ell(r)$ and proceed to discuss Born and improved Born approximations for $\tan\gamma_\ell(r)$. However, the generalizations are quite straightforward and no attempt shall be made here to deal with them.

III. The nonlocal Wronskian expressed in terms of phase-amplitude functions

If we differentiate the phase amplitude expression for $u_\ell(r)$ given by eq. (2-29), we have

$$u'_\ell(r) = \{\alpha'_\ell(r)\hat{D}_\ell(kr) + \alpha_\ell(r)\hat{D}'_\ell(kr)\}\sin[\hat{\delta}'_\ell(kr) + \delta_\ell(r)] + \alpha_\ell(r)\hat{D}_\ell(kr)\{\hat{\delta}'_\ell(kr) + \delta'_\ell(r)\}\cos[\hat{\delta}'_\ell(kr) + \delta_\ell(r)]. \quad (3-51)$$

Using eq. (2-31) to express $\alpha'_\ell(r)$ in terms of $\delta'_\ell(r)$, we find that the terms containing $\delta'_\ell(r)$ cancel and we can write

$$u'_\ell(r) = \frac{\hat{D}'_\ell(kr)}{\hat{D}_\ell(kr)} u_\ell(r) + \alpha_\ell(r)\hat{D}_\ell(kr)\hat{\delta}'_\ell(kr)\cos[\hat{\delta}'_\ell(kr) + \delta_\ell(r)]. \quad (3-52)$$

In a similar fashion we differentiate eq. (3-44) for $v_\ell(r)$ and use eq. (3-46) to obtain

$$v'_\ell(r) = \frac{\hat{D}'_\ell(kr)}{\hat{D}_\ell(kr)} v_\ell(r) - \beta_\ell(r) \hat{D}_\ell(kr) \hat{\delta}'_\ell(kr) \sin[\hat{\delta}_\ell(kr) + \gamma_\ell(r)]. \quad (3-53)$$

The nonlocal Wronskian then becomes

$$u'_\ell(r)v_\ell(r) - u_\ell(r)v'_\ell(r) = \alpha_\ell(r) \beta_\ell(r) \hat{D}_\ell^2(kr) \hat{\delta}'_\ell(kr) \cos[\delta_\ell(r) - \gamma_\ell(r)]. \quad (3-54)$$

From eq. (2-28) it may easily be verified that

$$\hat{D}_\ell^2(kr) \hat{\delta}'_\ell(kr) = k, \quad (3-55)$$

therefore

$$u'_\ell(r)v_\ell(r) - u_\ell(r)v'_\ell(r) = k \alpha_\ell(r) \beta_\ell(r) \cos[\delta_\ell(r) - \gamma_\ell(r)]. \quad (3-56)$$

It follows from eq. (3-15) that the interpolating function $f_\ell(r)$ is given by the simpler expression,

$$f_\ell^2(r) = \alpha_\ell(r) \beta_\ell(r) \cos[\delta_\ell(r) - \gamma_\ell(r)]. \quad (3-57)$$

Since the nature of the differential equations that $\alpha_\ell(r)$ and $\beta_\ell(r)$ satisfy ensure that both are positive and greater than zero, we conclude that Fiedeldey's method of obtaining an ELP is valid only for nonlocal potentials such that

$$|\delta_\ell(r) - \gamma_\ell(r)| < \frac{\pi}{2}.$$

If this condition is violated then we may expect poles to appear in the ELP. We must also ensure that $f_\ell(r)$ has the correct asymptotic behavior. If we take the limit of large r and use eq. (3-49), we find

$$\lim_{r \rightarrow \infty} f_\ell^2(r) = \alpha_\ell^2(\infty)$$

which is the correct normalization required by eq. (3-19).

IV. Interpolating function and ELP for a separable kernel

For the separable kernel given by eq. (2-45), we can immediately write down the solution to the irregular wave function from eq. (3-29).

$$v_\ell(r) = a_\ell \hat{j}_\ell(kr) - b_\ell \hat{n}_\ell(kr) - \frac{1}{B_\ell} \int_0^r [\hat{j}_\ell(kr) \hat{n}_\ell(ks) - \hat{j}_\ell(ks) \hat{n}_\ell(kr)] g_\ell(s) ds \quad (3-60)$$

where the constants B_ℓ are defined by

$$\frac{k}{B_\ell} = \int_0^\infty g_\ell(r) v_\ell(r) dr . \quad (3-61)$$

According to the prescription given by eqs. (3-31), (3-38), and (3-41) we have

$$\tan \gamma_\ell(r) = \frac{-a_\ell B_\ell + \int_0^r \hat{n}_\ell(ks) g_\ell(s) ds}{b_\ell B_\ell - \int_0^r \hat{j}_\ell(ks) g_\ell(s) ds} \quad (3-62)$$

and

$$\beta_\ell^2(r) = \left(b_\ell - \frac{1}{B_\ell} \int_0^r \hat{j}_\ell(ks) g_\ell(s) ds \right)^2 + \left(-a_\ell + \frac{1}{B_\ell} \int_0^r \hat{n}_\ell(ks) g_\ell(s) ds \right)^2 . \quad (3-63)$$

For convenience we define the functions

$$\begin{aligned} s_\ell(r) &= \int_0^r \hat{j}_\ell(ks) g_\ell(s) ds , \\ c_\ell(r) &= - \int_0^r \hat{n}_\ell(ks) g_\ell(s) ds . \end{aligned} \quad (3-64)$$

We now impose the end value conditions in eq. (3-49).

Thus, referring to eqs. (2-48) and (3-62) and imposing the condition, $\tan\delta_\ell(\infty) = \tan\gamma_\ell(\infty)$, we have

$$\frac{a_\ell B_\ell + c_\ell(\infty)}{s_\ell(\infty) - b_\ell B_\ell} = \frac{-s_\ell(\infty)}{A_\ell + c_\ell(\infty)} . \quad (3-65)$$

The constant B_ℓ is determined using eqs. (3-60) and (3-61)

$$\begin{aligned} B_\ell [a_\ell s_\ell(\infty) + b_\ell c_\ell(\infty)] &= k + \int_0^\infty g_\ell(s) ds \\ &\cdot \int_0^s [\hat{j}_\ell(ks) \hat{n}_\ell(kt) - \hat{j}_\ell(kt) \hat{n}_\ell(ks)] g_\ell(t) dt . \end{aligned}$$

However using eq. (2-50) this may be written as

$$B_\ell = \frac{A_\ell s_\ell(\infty)}{a_\ell s_\ell(\infty) + b_\ell c_\ell(\infty)} . \quad (3-66)$$

The nonlocal Wronskian (eq. (3-15)) may be calculated directly by using eq. (2-5) and eq. (3-60) for the non-local wave functions. The result is

$$f_\ell^2(r) = b_\ell - \frac{s_\ell(r)}{B_\ell} + \frac{a_\ell s_\ell(r) + b_\ell c_\ell(r)}{A_\ell} , \quad (3-67)$$

and thus

$$f_\ell^2(\infty) = b_\ell - \frac{s_\ell(\infty)}{B_\ell} + \frac{a_\ell s_\ell(\infty) + b_\ell c_\ell(\infty)}{A_\ell} .$$

Making use of eq. (3-66) in eq. (3-67) we have finally

$$f_\ell^2(\infty) = b_\ell . \quad (3-68)$$

Now we make use of the end value condition, $f_\ell^2(\infty) = \alpha_\ell^2(\infty)$, to obtain

$$b_\ell = 1 + \frac{2}{A_\ell} c_\ell(\infty) + \frac{c_\ell^2(\infty) + s_\ell^2(\infty)}{A_\ell^2} . \quad (3-69)$$

The above equation follows from eq. (2-49) for $\alpha_\ell^2(r)$.

Substituting for b_ℓ into eqs. (3-66) and (3-67) we also have

$$\frac{1}{B_\ell} = \frac{1}{A_\ell} \left(\frac{c_\ell(\infty)}{s_\ell(\infty)} \right) + \left(\frac{s_\ell^2(\infty) + c_\ell^2(\infty)}{s_\ell(\infty)} \right) \frac{1}{A_\ell^2} \quad (3-70)$$

and

$$a_\ell = \frac{1}{A_\ell} \left(\frac{s_\ell^2(\infty) - c_\ell^2(\infty)}{s_\ell(\infty)} \right) - \frac{c_\ell(\infty) (c_\ell^2(\infty) + s_\ell^2(\infty))}{s_\ell(\infty)} \frac{1}{A_\ell^2} . \quad (3-71)$$

The interpolating function and the ELP are then given by

$$f_\ell^2(r) = b_\ell \left(1 + \frac{s_\ell(\infty) c_\ell(r) - c_\ell(\infty) s_\ell(r)}{A_\ell s_\ell(\infty)} \right)$$

and

$$V_\ell(r) = -\frac{f_\ell''(r)}{f_\ell(r)} + 2 \left(\frac{f_\ell'(r)}{f_\ell(r)} \right)^2 + \frac{b_\ell}{f_\ell^2(r)} [c_\ell(\infty) \hat{j}_\ell'(kr) + s_\ell(\infty) \hat{n}_\ell'(kr)] \\ \cdot \frac{g_\ell(r)}{s_\ell(\infty)} .$$

CHAPTER 4

ANALYSIS OF A NONLOCAL KERNEL

We consider the class of nonlocal kernels that satisfy the differential equation

$$\left\{ \frac{d^2}{dr^2} - \beta^2 - \frac{\ell(\ell+1)}{r^2} \right\} K_\ell(r, r') w^{-\frac{1}{2}}(r) = -\lambda \beta^2 w^{\frac{1}{2}}(r') \delta(r-r') \quad (4-1)$$

with homogeneous boundary conditions

$$K_\ell(0, r') = K_\ell(\infty, r') = 0 . \quad (4-2)$$

The solutions to eq. (4-1) are given by

$$K_\ell(r, r') = \lambda \beta w^{\frac{1}{2}}(r) w^{\frac{1}{2}}(r') \hat{j}_\ell(i\beta r_<) \hat{h}_\ell^{(1)}(i\beta r_>) , \quad (4-3)$$

where $w(r)$ is a real short range function characteristic of a physical kernel and $\hat{h}_\ell^{(1)}(z)$ is the Riccati-Hankel function of the first kind defined by

$$\hat{h}_\ell^{(1)}(z) = \hat{j}_\ell(z) + i \hat{\eta}_\ell(z) .$$

This example is a special case of a more general class of nonlocal kernels considered by Razavy⁽⁸⁾ that can be obtained from the Green functions of second order differential equations. The kernel is physically realistic in the sense that the parameter λ measures

the strength of the interaction while $\frac{1}{\beta}$ is a measure of the range of nonlocality of the potential. That is; if we take the limit as $\frac{1}{\beta} \rightarrow 0$, we have from eq. (4-1)

$$K_\ell(r, r') = \lambda w(r) \delta(r-r')$$

and the interaction becomes local.

The choice of the "shape function" $w(r)$ is restricted by the boundary conditions imposed upon the kernel (eq. (4-2)). In addition, we must ensure the physical convergence of the phase-amplitude equations, (eqs. (2-30), (2-31)). Using the expansions of the Riccati-Bessel functions asymptotically and at the origin, it is easy to show that these requirements are met if we demand that $w(r)$ diverge less than $\frac{1}{r^2}$ at the origin and vanish faster than $\frac{1}{r}$ at infinity.

I. Nonlocal and momentum dependent potential

The nonlocal potential $V(\underline{r}, \underline{r}')$ is easily obtained by substituting the expression for the kernel (eq. (4-3)) into eq. (1-3) and using the expansion⁽⁹⁾

$$\frac{e^{-\beta|\underline{r}-\underline{r}'|}}{\beta|\underline{r}-\underline{r}'|} = \frac{1}{\beta^2 rr'} \sum_{\ell=0}^{\infty} (2\ell+1) \hat{j}_\ell(i\beta r_<) \hat{h}_\ell^{(1)}(i\beta r_>) P_\ell\left(\frac{\underline{r} \cdot \underline{r}'}{rr'}\right) .$$

We have

$$\frac{M}{\hbar^2} V(\underline{r}, \underline{r}') = \frac{\lambda \beta^2}{4\pi} w^{\frac{1}{2}}(r) w^{\frac{1}{2}}(r') \frac{e^{-\beta|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} . \quad (4-4)$$

Since the Yukawa term in eq. (4-4) satisfies

$$(\nabla^2 - \beta^2) \frac{e^{-\beta|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} = -4\pi\delta(\underline{r}-\underline{r}'),$$

then

$$(\nabla^2 - \beta^2) \frac{M}{h^2} \int V(\underline{r}, \underline{r}') w^{\frac{1}{2}}(\underline{r}) \psi(\underline{r}') d^3 \underline{r}' = -\lambda \beta^2 w^{\frac{1}{2}}(\underline{r}) \psi(\underline{r}),$$

or formally

$$\frac{M}{h^2} \int V(\underline{r}, \underline{r}') \psi(\underline{r}') d^3 \underline{r}' = \lambda \beta^2 w^{\frac{1}{2}}(\underline{r}) (\beta^2 - \nabla^2)^{-1} w^{\frac{1}{2}}(\underline{r}) \psi(\underline{r}).$$

The nonlocal Schrödinger equation (eq. (1-1)) becomes

$$(\nabla^2 + k^2) \psi(\underline{r}) = \lambda w^{\frac{1}{2}}(\underline{r}) \left(1 - \frac{\nabla^2}{\beta^2}\right)^{-1} w^{\frac{1}{2}}(\underline{r}) \psi(\underline{r}), \quad (4-5)$$

and the momentum dependent potential defined in eq. (1-7) is

$$U(\underline{r}, \underline{p}) = \lambda w^{\frac{1}{2}}(\underline{r}) \left(1 + \frac{\underline{p}^2}{\beta^2 h^2}\right)^{-1} w^{\frac{1}{2}}(\underline{r}).$$

II. The effective mass approximation

If one assumes that the nonlocality is small, that is, $\beta^2 \gg |\lambda w(\underline{r})|$ and $\beta^2 \gg k^2$, then one may expand the operator $(1 - \nabla^2/\beta^2)^{-1}$ to first order in $1/\beta^2$

$$\left(1 - \frac{\nabla^2}{\beta^2}\right)^{-1} \approx 1 + \frac{\nabla^2}{\beta^2}.$$

Therefore eq. (4-5) becomes

$$(\nabla^2 + k^2)\psi(\underline{r}) = \lambda w(r)\psi(\underline{r}) + \frac{\lambda}{\beta^2} w^{\frac{1}{2}}(r) \nabla^2 w^{\frac{1}{2}}(r) \psi(\underline{r}). \quad (4-6)$$

With the aid of the operator identity (here primes refer to differentiation with respect to r)

$$w^{\frac{1}{2}}(r) \{ \nabla^2 w^{\frac{1}{2}}(r) - w^{\frac{1}{2}}(r) \nabla^2 \} = \frac{1}{2} \{ w''(r) - \frac{[w'(r)]^2}{2w(r)} \} + \\ + w'(r) \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) ,$$

eq. (4-6) may be rewritten as

$$\nabla \cdot \left\{ \left(1 - \frac{\lambda w(r)}{\beta^2} \right) \nabla \psi(\underline{r}) \right\} + k^2 \psi(\underline{r}) \\ = \lambda \left\{ w(r) - \frac{1}{2\beta^2} \left[\frac{[w'(r)]^2}{2w(r)} - w''(r) - \frac{2w'(r)}{r} \right] \right\} \psi(\underline{r}) . \quad (4-7)$$

The above equation may be regarded as a generalization of the ordinary local Schrödinger equation with the mass M replaced by an effective mass

$$M^* = \frac{M}{1 - \frac{\lambda w(r)}{\beta^2}}$$

which is a function of position. Eq. (4-7) becomes

$$-\hbar^2 \left(\nabla \frac{1}{M^*(r)} \nabla \right) \psi(\underline{r}) + \frac{\hbar^2 \lambda}{M} \left\{ w(r) - \frac{1}{2\beta^2} \left[\frac{[w'(r)]^2}{2w(r)} - w''(r) - \frac{2w'(r)}{r} \right] \right\} \psi(\underline{r}) = E \psi(\underline{r}) . \quad (4-8)$$

This is the well known effective mass approximation of Frahn and Lemmer⁽¹⁰⁾.

III. Equivalent local potential in the effective mass approximation

It is possible to obtain an ELP in the effective mass approximation by making the transformation

$$\psi(\underline{r}) = \left(1 - \frac{\lambda w(r)}{\beta^2}\right)^{-\frac{1}{2}} \phi(\underline{r}) . \quad (4-9)$$

Here, $\phi(\underline{r})$ is the equivalent local wave function, for, if we use eq. (4-9) directly in eq. (4-7), we find that $\psi(\underline{r})$ satisfies the Schrödinger equation

$$\nabla^2 \phi(\underline{r}) + \{k^2 - v(r)\} \phi(\underline{r}) = 0 .$$

$v(r)$ is the energy dependent ELP given by

$$v(r) = \lambda \left\{ w(r) \left(1 - \frac{k^2}{\beta^2}\right) \left(1 - \frac{\lambda w(r)}{\beta^2}\right) - \frac{[w'(r)]^2}{4\beta^2 w(r)} \right\} \left(1 - \frac{\lambda w(r)}{\beta^2}\right)^{-2} . \quad (4-10)$$

The difference $\lambda w(r) - v(r)$ may be expressed as

$$\begin{aligned} \lambda w(r) - v(r) &= \lambda \left\{ \frac{w(r)}{\beta^2} \left(1 - \frac{\lambda w(r)}{\beta^2}\right) (k^2 - \lambda w(r)) + \frac{[w'(r)]^2}{4\beta^2 w(r)} \right\} \times \\ &\quad \times \left(1 - \frac{\lambda w(r)}{\beta^2}\right)^{-2} . \quad (4-11) \end{aligned}$$

We note that the ELP becomes $w(r)$ when the range of non-locality $(\frac{1}{\beta})$ is zero (as required). Since $w(r)$ is a positive shape function the right hand side of eq. (4-11) is negative for attractive potentials ($\lambda < 0$). In this case, the behavior of eq. (4-11) implies that for fixed $w(r)$, the average strength of the interaction decreases as a) the range of nonlocality $(\frac{1}{\beta})$ increases and b) the scattering energy increases.

IV. The nonlocal square well

If one multiplies the radial equation for nonlocal scattering (eq. (2-1)) by $w^{-\frac{1}{2}}(r)$, the differential equation for the kernel (eq. (4-1)) may be used to obtain a fourth order differential equation for the nonlocal wave function

$$\begin{aligned} & \left\{ \frac{d^2}{dr^2} - \beta^2 - \frac{\ell(\ell+1)}{r^2} \right\} w^{-\frac{1}{2}}(r) \left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} u_\ell(r) \\ &= -\lambda \beta^2 w^{\frac{1}{2}}(r) u_\ell(r) . \end{aligned} \quad (4-12)$$

One of the few cases for which analytic solutions to eq. (4-13) exist is that of the square well of range b

$$w(r) = \begin{cases} 1 & r < b \\ 0 & r > b \end{cases}$$

Since the local potential well is known to be a rough approximation to the actual interaction between nucleons, it is worthwhile to study this more sophisticated non-local model. De La Cruz et al⁽¹¹⁾ attempted to fit the experimental phase shifts for proton-proton scattering in the 1S state using an attractive nonlocal square well in a range intermediate between a repulsive hard core and an attractive Eckart potential. They deduced a set of parameters that was effective in fitting the phase shifts for laboratory energies in the range 20-340 Mev.

We shall study only the pure nonlocal square well here with the specific purpose of determining its equivalent local wave function and potential. For the s-state, eq. (4-12) reduces to

$$\left(\frac{d^2}{dr^2} - \beta^2 \right) \left(\frac{d^2}{dr^2} + k^2 \right) u(r) = -\lambda \beta^2 u(r), \quad (r < b) . \quad (4-13)$$

The solutions to eq. (4-13) are expressed as linear combinations of the functions $\sinh \mu r$, $\cosh \mu r$, $\sin v r$ and $\cos v r$, where

$$\mu^2 = \frac{1}{2} \{ \beta^2 - k^2 + \sqrt{(\beta^2 + k^2)^2 - 4\beta^2 \lambda} \} \quad (4-14)$$

$$v^2 = \frac{1}{2} \{ k^2 - \beta^2 + \sqrt{(\beta^2 + k^2)^2 - 4\beta^2 \lambda} \} .$$

For attractive potentials ($\lambda < 0$), μ^2 and ν^2 are always positive.

The wave function is uniquely determined by the boundary conditions

$$u(r) \xrightarrow[r \rightarrow 0]{} kr$$

$$u''(0) + k^2 u(0) = 0$$

$$u''(b) + k^2 u(b) = 0$$

where the last two equations are a consequence of eq.(4-2). The wave function is written as

$$u(r) = \frac{k \sin \nu b \sinh \mu b}{\nu(\mu^2 + k^2) \sinh \mu b + \mu(\nu^2 - k^2) \sin \nu b} \times \\ \times \{(\nu^2 - k^2) \frac{\sinh \mu r}{\sinh \mu b} + (\mu^2 + k^2) \frac{\sin \nu r}{\sin \nu b}\} . \quad (4-15)$$

The phase and amplitude functions are then immediately determined from the equations

$$\alpha(r) \sin[kr + \delta(r)] = u(r)$$

$$k \operatorname{ctn} [kr + \delta(r)] = \frac{1}{u(r)} \frac{du(r)}{dr} .$$

The irregular solution to the nonlocal wave function, which we require in order to compute the equivalent quantities, is given by

$$v(r) = A \sinh \mu r + B \cosh \mu r + C \sin \nu r + D \cos \nu r .$$

(4-16)

The boundary conditions

$$v''(0) + k^2 v(0) = 0$$

$$v''(b) + k^2 v(b) = 0$$

imply

$$D = \frac{\mu^2 + k^2}{\nu^2 - k^2} B$$

$$A = \left(\frac{\cos \nu b - \cosh \mu b}{\sinh \mu b} \right) B + \left(\frac{\nu^2 - k^2}{\mu^2 + k^2} \right) \frac{\sin \nu b}{\sinh \mu b} C . \quad (4-17)$$

The constants B and C are determined according to the prescription given in Section II of Chapter 3. That is, we determine the irregular phase and amplitude functions from the equations

$$\beta(r) \cos [kr + \gamma(r)] = v(r)$$

$$-k \tan [kr + \gamma(r)] = \frac{v'(r)}{v(r)} .$$

The end value conditions, $\beta(b) = \alpha(b)$, and $\gamma(b) = \delta(b)$, then determine the constants B and C. However, it is unnecessary to do this calculation, for if we directly use eqs. (4-15), (4-16) and (4-17) to evaluate the nonlocal Wronskian, we obtain

$$kf^2(r) = u'(r)v(r) - u(r)v'(r) \quad (4-18)$$

$$= kB \left(\frac{\mu^2 + \nu^2}{\nu^2 - k^2} \right) \left\{ 1 + \frac{(\nu^2 - k^2)(\mu^2 + k^2)T(r)}{(\mu^2 + \nu^2) [\nu(\mu^2 + k^2) \sinh \mu b + \mu(\nu^2 - k^2) \sin \nu b]} \right\}$$

where the function $T(r)$ is given by

$$T(r) = \nu(\sinh \mu r \cos \nu(b-r) + \cos \nu r \sinh \mu(b-r) - \sinh \mu b) \quad (4-19)$$

$$+ \mu(\sin \nu r \cosh \mu(b-r) + \cosh \mu r \sin \nu(b-r) - \sin \nu b) .$$

Since $T(0) = T(b) = 0$, we have

$$f^2(0) = f^2(b) = \frac{\mu^2 + \nu^2}{\nu^2 - k^2} B \quad (4-20)$$

and from eq. (4-14)

$$(\nu^2 - k^2)(\mu^2 + k^2) = -\lambda \beta^2 .$$

Therefore eq. (4-18) can be written as

$$\left(\frac{f(r)}{f(b)} \right)^2 = 1 - \frac{\lambda \beta^2 T(r)}{(\mu^2 + \nu^2) [\nu(\mu^2 + k^2) \sinh \mu b + \mu(\nu^2 - k^2) \sin \nu b]} . \quad (4-21)$$

V. Properties of the interpolating function $f(r)$

The function $T(r)$ defined by eq. (4-19) is symmetrical about $r=b/2$. Also the upper and lower bounds on $T(r)$

are quite easily obtained. We can rewrite eq. (4-19) in two different ways

$$T(r) = (v \sinh \mu r - \mu \sin v r) (\cos v(b-r) - \cosh \mu(b-r)) + (v \sinh \mu(b-r) - \mu \sin v(b-r)) (\cos v r - \cosh \mu r) \quad (4-22)$$

and

$$T(r) + 2(v \sinh \mu b + \mu \sin v b) = (v \sinh \mu r + \mu \sin v r) (\cos v(b-r) + \cosh \mu(b-r)) + (v \sinh \mu(b-r) + \mu \sin v(b-r)) (\cos v r + \cosh \mu r) \quad (4-23)$$

Equations (4-22) and (4-23) follow from eq. (4-19) upon replacing b by $r+b-r$ and expanding $\sin v b$ and $\sinh \mu b$. Because of the inequalities

$$\left. \begin{array}{l} v \sinh \mu x \geq \mu \sin v x \\ \cosh \mu x \geq \cos v x \end{array} \right\} \quad (0 \leq x \leq b) ,$$

we can see by inspection that the right hand side of eq. (4-22) is always negative, while the right hand side of eq. (4-23) is always positive. It follows that

$$-2(v \sinh \mu b + \mu \sin v b) \leq T(r) \leq 0 \quad (4-24)$$

Eq. (4-24) can be used directly in eq. (4-21) to obtain bounds on the ratio $f(r)/f(b)$. With the aid of eq. (4-14) we have

$$\left(\frac{\beta^2 + k^2}{\mu^2 + \nu^2}\right) \cdot \frac{\nu(\mu^2 + k^2) \sinh \mu b - \mu(\nu^2 - k^2) \sin \nu b}{\nu(\mu^2 + k^2) \sinh \mu b + \mu(\nu^2 - k^2) \sin \nu b} \leq \left(\frac{f(r)}{f(b)}\right)^2 \leq 1 \quad . \quad (4-25)$$

It is convenient to define a renormalized nonlocal wave function $\bar{u}(r)$ by $\bar{u}(r) = u(r)/f(b)$. Since $u(r) = f(r)x(r)$ we have

$$\frac{\bar{u}(r)}{x(r)} = \frac{f(r)}{f(b)} \xrightarrow{r \rightarrow b} 1 \quad . \quad (4-26)$$

There are two observations of importance that can be made regarding eq. (4-25). The first is the fact that the lower limit of $(f(r)/f(b))^2$ is always greater than zero and therefore no poles can appear in the ELP. For our particular example, Fiedeldey's method of constructing an ELP is entirely consistent and the assumption that $f(r)$ be greater than zero is not necessary.

The second observation is that the ratio $f(r)/f(b) = \bar{u}(r)/x(r)$ is always less than unity within the potential well and therefore the nonlocal wave function $\bar{u}(r)$ is always smaller than the equivalent local wave function $x(r)$ there. This result supports an argument in a paper by Austern⁽¹²⁾ where an approximate procedure is used to construct the equivalent local wave function. He points out that, in general, an attractive nonlocal potential equivalent to a local potential (in the sense that they

give rise to the same asymptotic wave functions for all energies) produces a wave function that is always smaller inside the region of the potential than the equivalent local wave function.

VI. The ELP for the nonlocal square well

The equivalent local potential for the square well is determined from the expression

$$v(r) = - \frac{f''(r)}{f(r)} + 2 \left(\frac{f'(r)}{f(r)} \right)^2 + \frac{1}{kf^2(r)} \int_0^b K(r,s) [u'(r)v(s) - v'(r)u(s)] ds .$$

Because $u(r)$ and $v(r)$ both satisfy the nonlocal wave equation (eq. (2-1)), the integral in the above equation can be rewritten in terms of differentials of $u(r)$ and $v(r)$. Making use of the nonlocal Wronskian (eq. (3-15)), we obtain

$$v(r) = - \frac{f''(r)}{f(r)} + 2 \left(\frac{f'(r)}{f(r)} \right)^2 + \frac{u'(r)v''(r) - v'(r)u''(r)}{kf^2(r)} + k^2 . \quad (4-27)$$

The first two terms in the above equation are evaluated by differentiating eq. (4-21). The last two terms are

evaluated using eqs. (4-15), (4-16), (4-17) and (4-20).

After a somewhat tedious calculation we obtain

$$\frac{v(r)}{\lambda} = \beta^2 \frac{(\mu S_2(r) - v S_1(r))}{2Q(r)} + \frac{3}{4} \lambda \beta^4 \frac{S_3^2(r)}{Q^2(r)} + \frac{\beta^2}{(\mu^2 + v^2) Q(r)} \{ v(k^2 + \mu^2) (\sinh \mu b - S_1(r)) + \mu(k^2 - v^2) (\sinh v b - S_2(r)) \}, \quad (4-28)$$

where for convenience we have defined

$$Q(r) = \left(\frac{f(r)}{f(b)} \right)^2 [v(\mu^2 + k^2) \sinh \mu b + \mu(v^2 - k^2) \sinh v b],$$

$$S_1(r) = \sinh \mu r \cos v(b-r) + \cos v r \sinh \mu(b-r), \quad (4-29)$$

$$S_2(r) = \sinh v r \cosh \mu(b-r) + \cosh \mu r \sinh v(b-r),$$

$$S_3(r) = \sinh \mu r \sinh v(b-r) - \sinh v r \sinh \mu(b-r).$$

For large β ($\beta^2 \gg k^2, |\lambda|$), the expressions for μ^2 and v^2 in eq. (4-14) may be expanded in powers of $1/\beta^2$.

To lowest order we have

$$\mu^2 \approx \beta^2 - \lambda$$

$$v^2 \approx k^2 - \lambda,$$

and from eq. (4-22) we see that

$$T(r) \approx -\sqrt{k^2 - \lambda} \sinh \beta b.$$

Therefore, the ratio of nonlocal to local wave functions becomes

$$\left(\frac{\bar{u}(r)}{x(r)}\right)^2 = \left(\frac{f(r)}{f(b)}\right)^2 \approx 1 + \frac{\lambda}{\beta^2} .$$

However, this ratio is equal to that obtained in the effective mass approximation (eq. (4-9)), where we had

$$\left(\frac{\psi(r)}{\phi(r)}\right)^2 = \left(1 - \frac{\lambda}{\beta^2}\right)^{-1} \approx 1 + \frac{\lambda}{\beta^2} .$$

To evaluate the ELP for large β , we note that the first term in eq. (4-28) is proportional to $T''(r)$ while the second is proportional to $[T'(r)]^2$. Because the function $T(r)$ becomes a constant at large β , these terms effectively vanish. Specifically, it can be shown that they vanish faster than $1/\beta^2$ for large β . For the remaining term in eq. (4-28) we can write

$$\begin{aligned} & v(k^2 + \mu^2) (\sinh \mu b - S_1(r) + \mu(k^2 - v^2) (\sin v b - S_2(r))) \\ &= -k^2 T(r) - \mu v [(\mu \sinh \mu r + v \sin v r) (\cos v(b-r) - \cosh \mu(b-r)) \\ & \quad + (v \sin v(b-r) + \mu \sinh \mu(b-r)) (\cos v r - \cosh \mu r)] \\ & \approx \sqrt{k^2 - \lambda} (\beta^2 + k^2 - \lambda) \sinh \beta b . \end{aligned}$$

Now, since

$$Q(r) \approx \beta^2 \left(1 + \frac{\lambda}{\beta^2}\right) \left(1 + \frac{k^2 - \lambda}{\beta^2}\right) \sqrt{k^2 - \lambda} \sinh \beta b ,$$

we have

$$V(r) \approx \frac{\lambda}{\left(1 + \frac{\lambda}{\beta^2}\right) \left(1 + \frac{k^2 - 2\lambda}{\beta^2}\right)} \approx \lambda \left(1 + \frac{\lambda - k^2}{\beta^2}\right) .$$

This is just the equivalent local potential in the effective mass approximation because from eq. (4-10) we have

$$[V(r)]_{\text{eff.mass.approx.}} = \frac{\lambda \left(1 - \frac{k^2}{\beta^2}\right)}{\left(1 - \frac{\lambda}{\beta^2}\right)} \approx \lambda \left(1 + \frac{\lambda - k^2}{\beta^2}\right) .$$

Therefore, within the range of the interaction ($0 < r < b$), we have verified that the ELP obtained from Fiedeldey's method matches that of the effective mass approximation.

The ELP for the nonlocal square well (eq. (4-28)) is a rather formidable function of radial distance, scattering energy, nonlocality and well depth and range. No attempt is made here to study its behavior analytically. A numerical calculation of the ratio $V(r)/\lambda$ was made for the purpose of extracting some of the qualitative features of the ELP. The range of values studied was

$$1 < E < 256 \text{ Mev.}$$

$$-640 < v_0 < -10 \text{ MeV.}$$

$$\frac{1}{4} < \beta < 16 \text{ (fermi)}^{-1}$$

$$1 < b < 5 \text{ fermi} ,$$

where $v_0 = \lambda \hbar^2 / M$, $E = \hbar^2 k^2 / M$ and M is the nucleon mass.

Figures (1.-3.) offer in part some idea of the behavior of the ELP. It was generally found that the ratio

$v(r)/\lambda$

- a) is always less than unity and decreases as the range of nonlocality ($1/\beta$) increases (Fig. 1.),
- b) decreases as the scattering energy increases (Fig. 2.),
- c) decreases as the well depth $|\lambda|$ increases (Fig. 3.).

An obvious disadvantage of the equivalent local potentials in figures (1.-3.) is the fact that they have cusps of finite height and width at $r \gtrsim 0$ and $r \lesssim b$. This irregular behavior is a consequence of our choosing as a shape function $w(r)$, the unit step function, which has a discontinuity at $r=b$. This discontinuity makes the ratio $f''(r)/f(r)$ in eq. (4-27) to be finite at $r=0$ and $r=b$ and thus causes the cusp. Of course, as the interaction becomes local ($1/\beta \rightarrow 0$), the width of the cusp vanishes and the step function is retained as the ELP. This suggests that Fiedeldey's method should more appreciably be applied to shape functions that decay smoothly to zero. Unfortunately it would be extremely difficult to analyse such nonlocal potentials analytically.

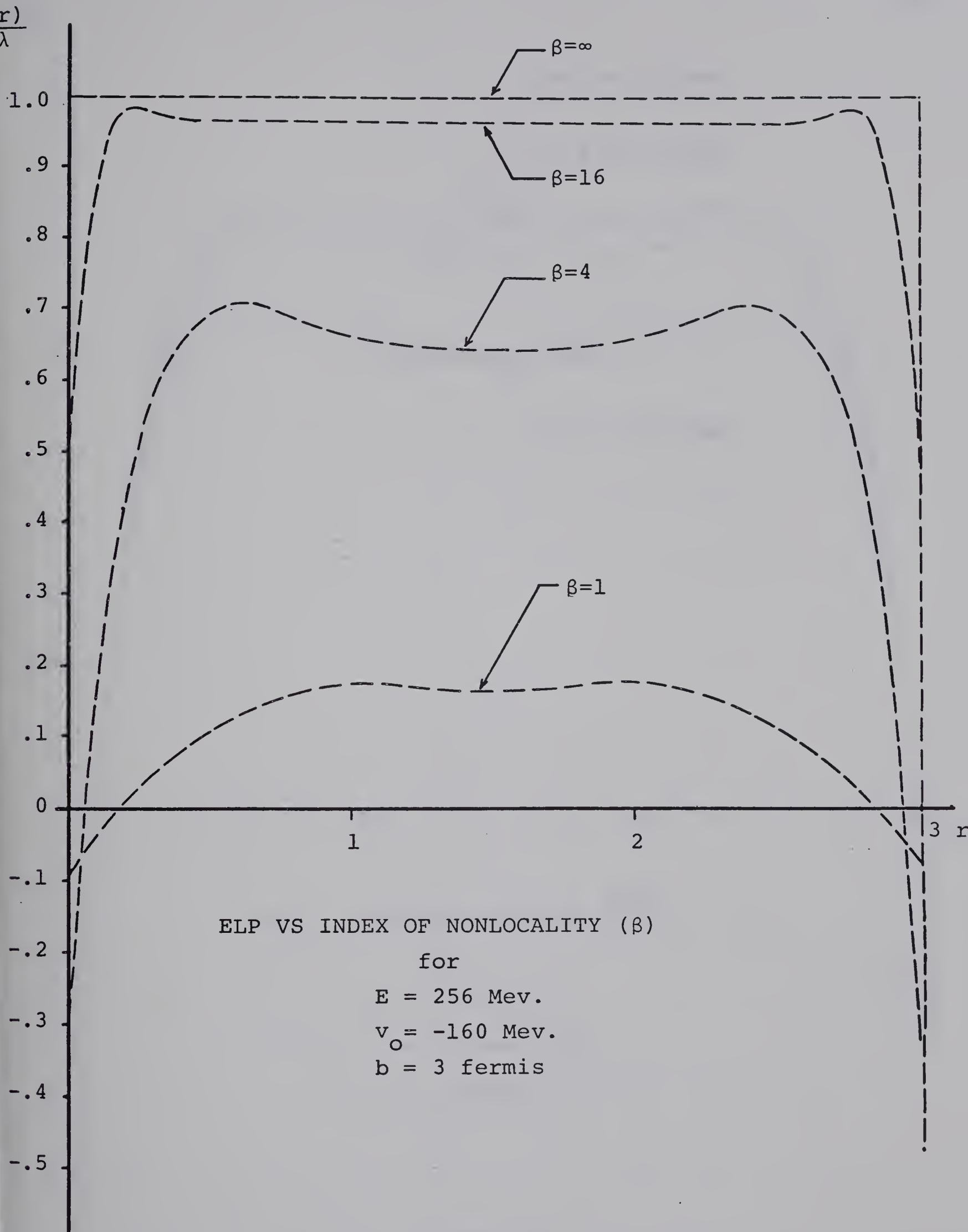
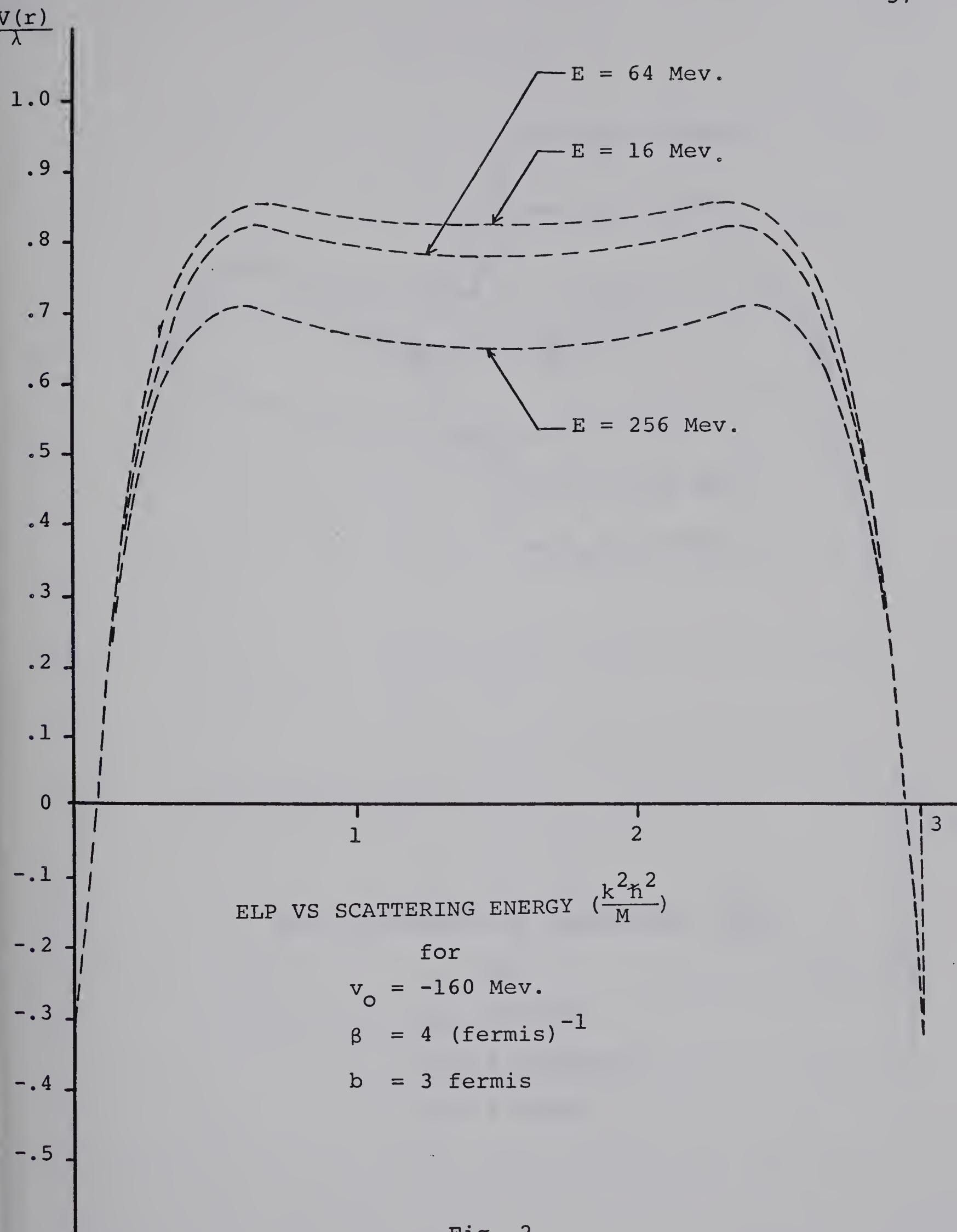


Fig. 1.



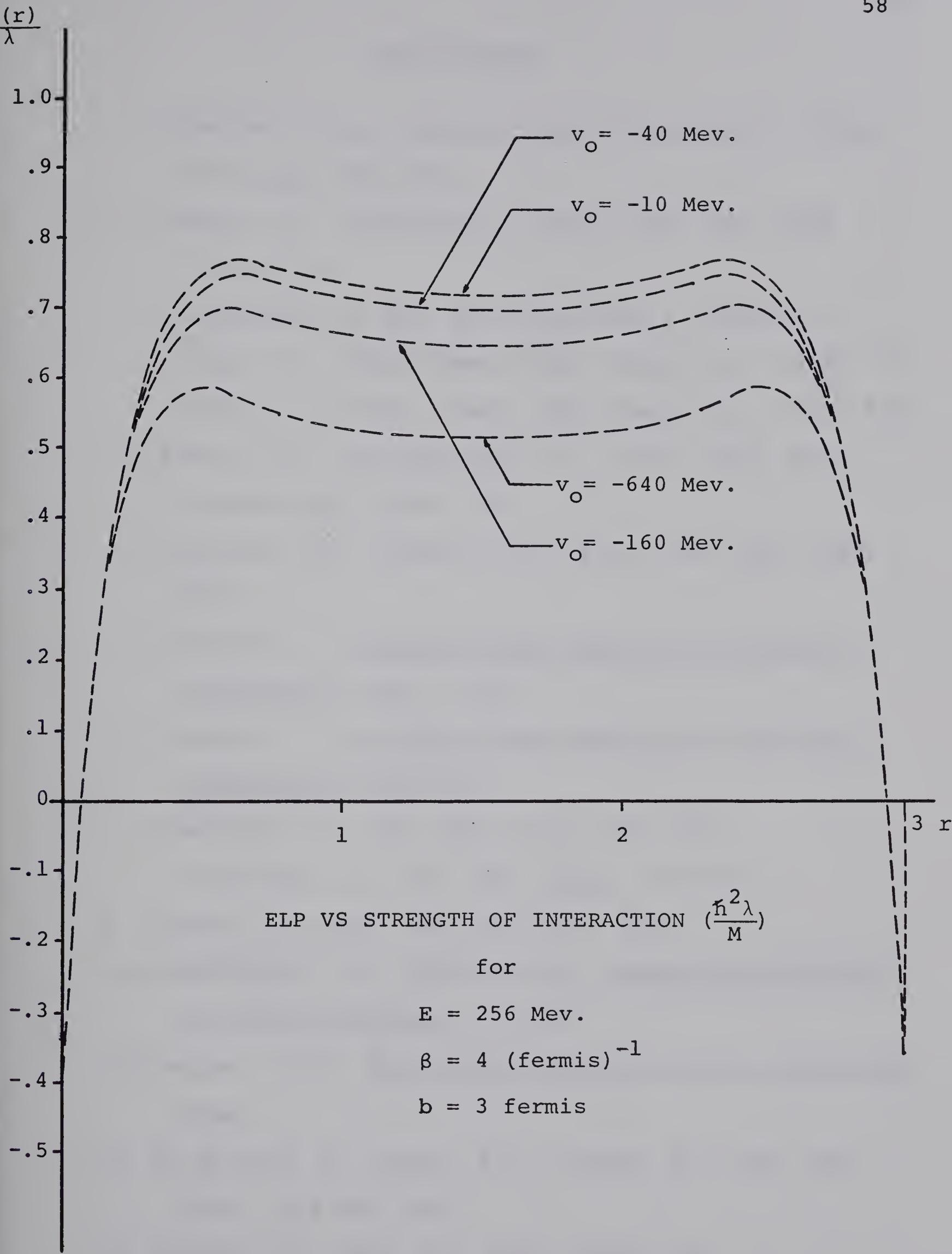


Fig. 3.

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B29963